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ON A PROBLEM OF FIXING THE LEVEL OF INDEPENDENT
VARIABLES IN A LINEAR REGRESSION FUNCTION

Kei Takeuchi

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ABSTRACT

Suppose that a linear regression model

$$Y = \beta'x + U$$

is given. We want to fix x so as to make $E(Y) = \beta'x$ as near to some prescribed level c as possible.

Asymptotic consideration leads to a solution of the type

$$\hat{x} = \frac{c M \hat{\beta}}{\hat{\beta}' M \hat{\beta} + k \hat{\sigma}^2} ,$$

where $\hat{\beta}$ is the least square estimator of β , and $\hat{\sigma}^2$ is the unbiased estimator of $\sigma^2 = V(U)$.

Under the assumption of normality for the distribution of U , an exact formula for the first two moments of the error $\beta'\hat{x}$ is given, and by expanding the formula for the mean square error, it is recommended that k be chosen to be equal to $\max(5-p, 0)$, where p is the dimension of the x vector.

0. Introduction

In the practical application of statistical techniques to industrial or business problems, it often happens that we want to find an appropriate level of the "policy" or "control" variable (or vector of control variables) z , so that some quantity Y may be as near to some prescribed level c as possible.

If the relation of Y and Z is described by a stochastic model which includes some unknown parameter θ , and if we have a set of observations on Y and $\hat{Z}$, then a problem of mathematical statistics, or of statistical decision, can be formulated. We want to find a "policy" function Z based on the observations so that the error may be stochastically as small as possible.

If $E(Y) = \eta = f(\theta, z)$ and if $Y - E(Y)$ is independent of the past observations, we want to make the mean square

$$E_{\theta}(c - f(\theta, \hat{Z}))^2$$

as small as possible.

Usually, if some estimator $\hat{\theta}$ of θ is given, we are inclined to determine z so as to satisfy $f(\hat{\theta}, \hat{z}) = c$. But there still remain two problems: (i) Generally, there are infinitely many combinations of values of z which satisfy the above. So we have to determine $\hat{z}$ in order to minimize the error due to the statistical error of $\hat{\theta}$. (ii) In some cases, it does not follow that determining $\hat{z}$ to satisfy the above is a good policy even if $\hat{\theta}$ is, in some sense, an optimum or efficient estimator.

Consequently, we need a more detailed analysis of the situation than the mere translation of estimation into some kind of decision theoretic language.

In most practical situations the model is given in a linear regression set-up, i.e.,

$$E(Y) = \beta'x.$$

We shall discuss the general problem in the asymptotic case, in Section 1, and then more detailed considerations will be given in the regression case in Sections 2-4, where exact small sample results are established.

We will discuss the problem in both non-Bayesian and Bayesian terms, and we will show that in some situations the Bayes solution with respect to pseudo density, usually adopted in regression situations as in Raiffa and Schlaifer [3], and also in Zellner and Chetty [4], is inadmissible at least asymptotically.

1. Asymptotic Considerations

Suppose that we are to make

$$(1.1) \quad \eta = f(\theta, z)$$

as near as possible to the prescribed constant c .

In (1.1) it is assumed that θ is a p -dimensional vector of unknown parameters, and z is a q -dimensional vector of "control" or "policy" variables which we can fix at any level in a set C . We assume that for any θ , there is at least one $z_0 \in C$ such that $f(\theta, z_0) = c$.

Suppose that we have a set of observations $\tilde{X} = (X_1 \dots X_n)$, with density function $p(\tilde{x}, \theta)$ with respect to some σ -finite measure μ . We can assume that our problem is to choose a measurable function $\hat{Z} = \hat{Z}(\tilde{X})$ from the space of the range of $\tilde{X}$ into C . We want to make the mean square error

$$(1.2) \quad E(f(\theta, \hat{Z}) - c)^2$$

as small as possible.

We shall assume the following.

(1) $f(\theta, z)$ is continuously differentiable with respect to θ and z .

It is difficult to derive general results for small sample set-ups, so that we shall consider the asymptotic case in which it is assumed that

(2) $\hat{Z}(\tilde{X})$ is distributed with mean $\bar{Z}(\theta)$ and variance-covariance matrix Σ of smaller order than 1.

Then

$$f(\theta, \hat{z}) - c = f(\theta, \bar{z}) - c + \left(\frac{\partial f}{\partial z} \right)'_{z=\bar{z}} (\hat{z} - \bar{z}) + R ,$$

where $\partial f / \partial z$ denotes the vector of partial derivatives and where R is stochastically of smaller order than the preceding terms. From (1.3) we have the formula for the asymptotic mean and asymptotic mean square of the error as

$$E(f(\theta, \hat{z}) - c) \sim f(\theta, \bar{z}) - c$$

$$E(f(\theta, \hat{z}) - c)^2 \sim (f(\theta, \bar{z}) - c)^2 + \left(\frac{\partial f}{\partial z} \right)' \sum \left(\frac{\partial f}{\partial z} \right).$$

The asymptotic mean or asymptotic variance may not be equal to the asymptotic values of the mean or variance, but we shall now consider the former as our criterion.

We shall say a function z is asymptotically consistent if

$$f(\theta, \bar{z}(\theta)) = c, \quad \text{for all } \theta,$$

and a function z which minimizes

$$\left(\frac{\partial f}{\partial z} \right)'_{z=\bar{z}(\theta)} \sum \left(\frac{\partial f}{\partial z} \right)$$

among all asymptotically consistent functions asymptotically best consistent or, for short, ABC.

We shall obtain some lower bound for (1.5) under the assumption that

(3) The density function $p(\tilde{x}, \theta)$ is continuously differentiable with respect to θ almost everywhere on $\tilde{x}$, and

$$\lim_{\Delta\theta \rightarrow \theta} \int \left\| \frac{p(\tilde{x}, \theta + \Delta\theta) - p(\tilde{x}, \theta)}{p(\tilde{x}, \theta) \Delta\theta} - \frac{p_{\theta}(\tilde{x}, \theta)}{p(\tilde{x}, \theta)} \right\|^2 p(\tilde{x}, \theta) d\mu = 0$$

where $p_{\theta}(\tilde{x}, \theta) = \partial p(\tilde{x}, \theta) / \partial \theta$.

Then, from (1.4),

$$\frac{\partial f}{\partial \theta} - \frac{\partial f}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \theta} = 0,$$

where $\partial \bar{z} / \partial \theta$ is the Jacobian matrix of derivatives. From the well known Cramer-Rao theorem,

$$t' \sum t \geq t' \left(\frac{\partial \bar{z}}{\partial \theta} \right)' I_{\theta} \left(\frac{\partial \bar{z}}{\partial \theta} \right) t,$$

where I_{θ} is the Fisher information matrix. From (1.6) and (1.7),

$$\left(\frac{\partial f}{\partial \bar{z}} \right)' \sum \left(\frac{\partial f}{\partial \bar{z}} \right)^* \geq \left(\frac{\partial f}{\partial \theta} \right)' I_{\theta}^{-1} \left(\frac{\partial f}{\partial \theta} \right)$$

is obtained.

Hence, a lower bound for the asymptotic variance of the asymptotically consistent function is given by

$$\inf_{\substack{\bar{z} \\ f(\theta, \bar{z})=c}} \left(\frac{\partial f}{\partial \theta} \right)' I^{-1} \left(\frac{\partial f}{\partial \theta} \right) \Big|_{z=\bar{z}}.$$

If a function $\hat{z}$ attains this lower bound asymptotically, we shall say that it is asymptotically efficient.

In practical situations, it often happens that there exists an estimator $\hat{\theta}$ which is asymptotically efficient in the sense that it is asymptotically unbiased and its asymptotic variance-covariance matrix is equal to I_{θ}^{-1} . Let us consider

a function $\hat{z}^* = \hat{z}(\hat{\theta})$ of $\hat{\theta}$, which is determined so as to minimize

$$\left(\frac{\partial f}{\partial \theta}\right)' \Big|_{z=\hat{z}} I_{\hat{\theta}}^{-1} \left(\frac{\partial f}{\partial \theta}\right)$$

under the condition that $f(\hat{\theta}, \hat{z}^*(\hat{\theta})) = c$. If we assume that

(4) $\hat{z}^*(\hat{\theta})$ is continuously differentiable with respect to θ , then we can expand

$$\begin{aligned} f(\theta, \hat{z}^*) &= f(\theta, \hat{z}^*(\theta)) = \left(\frac{\partial f}{\partial z}\right)' \Big|_{z=\hat{z}(\theta)} (\hat{z}^*(\theta) - \hat{z}(\theta)) + R \\ &= c + \left(\frac{\partial f}{\partial z}\right)' \left(\frac{\partial \hat{z}^*}{\partial \theta}\right) (\hat{\theta} - \theta) + R' \\ &= c - \left(\frac{\partial f}{\partial \theta}\right)' \Big|_{z=\hat{z}(\theta)} (\hat{\theta} - \theta) + R' , \end{aligned}$$

since the relation (1.6) must also hold for $\hat{z}^*$.

Hence,

$$E(f(\theta, \hat{z}^*) - c)^2 \sim \left(\frac{\partial f}{\partial \theta}\right)' \Big|_{z=\hat{z}(\theta)} I_{\theta}^{-1} \left(\frac{\partial f}{\partial \theta}\right) ,$$

which is actually equal to (1.8). Consequently, $\hat{z}^*$ is asymptotically equivalent to the ABC function, and we shall also call it an ABC function based on $\hat{\theta}$. In the usual situation where some regularity conditions hold true, we can take the maximum likelihood estimator for $\hat{\theta}$.

A different approach would be that of Bayes.

If we assume a prior distribution for θ , we can obtain a posterior distribution for θ , and we must simply determine z

so as to minimize the expectation of

$$(f(\theta, z) - c)^2$$

with respect to the posterior distribution of θ .

We shall denote by $\tilde{\theta}$ the random variable distributed according to the posterior distribution of θ . Then the problem is to minimize

$$(1.10) \quad E\{f(\tilde{\theta}, z) - c\}^2.$$

It often happens in an asymptotic situation (see Le Cam [2]), that $\tilde{\theta}$ is distributed with mean nearly equal to $\hat{\theta}$, and the variance covariance matrix $I\hat{\theta}^{-1}$, given $\hat{\theta}$, where $\hat{\theta}$ is the maximum likelihood estimator.

In that case,

$$(1.11) \quad E\{f(\tilde{\theta}, z) - c\}^2 \doteq (f(\hat{\theta}, z) - c)^2 + \left(\frac{\partial f}{\partial \theta}\right)'_{\theta=\hat{\theta}} I_{\hat{\theta}}^{-1} \left(\frac{\partial f}{\partial \theta}\right).$$

Hence the Bayes solutions is nearly equal to the value of z which minimizes (1.11). Therefore, we shall call a function z , which is determined to minimize the right side of (1.11), asymptotic Bayes or, for short, an AB function.

Since the second term will be small for some range of z , $f(\hat{\theta}, z) - c$ will be approximately equal to zero for the AB function; hence, it will be approximately equivalent to the ABC function based on $\hat{\theta}$.

In subsequent sections, we shall investigate the properties of the AMC or AB procedures for more specific cases in more detail.

2. The Linear Regression Case

Now we shall apply the discussion of the preceding section to the problem of linear regression.

Assume that

$$(2.1) \quad Y = \beta'x + U$$

where β and x are p -dimensional vectors of the parameters and the "control" variables, and the error U is assumed to be normally distributed. We want to fix the level of x in order that

$$E(Y) = f(\beta, x) = \beta'x$$

may be as near to a constant c as possible.

We assume that we have a set of independent observations of size n on x and Y , and suppose that we have the least square estimator $\hat{\beta}$ for β , and $\hat{\sigma}^2$ for σ^2 calculated from the observations. Since it is well known that $(\hat{\beta}, \hat{\sigma}^2)$ forms a sufficient statistic in this case, we can determine x based solely on $\hat{\beta}$ and $\hat{\sigma}^2$.

In this situation, it is well known that the information matrix for β is equal to the moment matrix M of x , divided by σ^2 , hence the ABC function is determined to minimize

$$x' M^{-1} x$$

under the condition that $\hat{\beta}'x = c$, and it is easily seen that it is given by

$$(2.2) \quad \hat{x} = \frac{c M \hat{\beta}}{\hat{\beta}' M \hat{\beta}},$$

and the asymptotic mean square error is given by

$$(2.3) \quad \frac{\sigma^2}{\beta' M \beta}.$$

It is a problem whether the asymptotic value above is really the limit of the mean square error. As will be shown in the next section, it is true for the case when $p \geq 3$, but if $p \leq 2$, the mean square error is always infinity.

The asymptotic Bayes solution is given by minimizing

$$(\hat{\beta}' x - c)^2 + x' M^{-1} x \hat{\sigma}^2,$$

the solution of which is given by

$$(2.3) \quad \hat{x} = (M^{-1} \hat{\sigma}^2 + \hat{\beta} \hat{\beta}')^{-1} c \cdot \hat{\beta} = \frac{c M \hat{\beta}}{\hat{\beta}' M \hat{\beta} + \hat{\sigma}^2}.$$

It should be remarked that the two solutions given in (2.1) and (2.3) differ only in the scale factor.

If we take as the prior density for β and σ the pseudo-density of the form

$$p(\tilde{\beta}, \tilde{\sigma}) \propto \frac{1}{\tilde{\sigma}}$$

then the posterior distribution of β , given $\hat{\beta}$ and $\hat{\sigma}^2$, is known to be a multivariate t-distribution ([1]). Hence, the posterior mean and posterior variance are equal to $\hat{\beta}$, and $q M^{-1} \hat{\sigma}^2 / (q-2)$, respectively, so that the Bayes solution is, as was shown by Zellner and Chetty [4],

$$(2.4) \quad \hat{x} = \frac{cM\hat{\beta}}{\hat{\beta}'M\hat{\beta} + \frac{q\hat{\sigma}^2}{q-2}}$$

which is nearly equivalent to (2.3) when q is large.

In some cases, the independent value x cannot take arbitrary values. Often it is required that some linear restriction

$$(2.5) \quad Ax = b$$

be satisfied. In (2.5) we assume that A is an $r \times p$ matrix of rank r ($r < p$) and b is a constant r -dimensional vector.

Then, for ABC x is determined to minimize $x'M^{-1}x$, under the condition $\hat{\beta}'x = c$ as well as (2.5), and it is given by

$$(2.6) \quad \hat{x} = M(A'\lambda + \mu\hat{\beta}) ,$$

where λ is a vector of r -dimensional Lagrangian multipliers, and μ is another Lagrangian multiplier. Both λ and μ are determined from

$$\hat{\beta}'MA'\lambda + \mu\hat{\beta}'M\hat{\beta} = c ,$$

$$AMA'\lambda + \mu AM\hat{\beta} = b ,$$

from which it is derived that

$$\begin{aligned} \lambda &= (AMA')^{-1}(b - \mu AM\hat{\beta}) ; \\ \mu &= \frac{c - \hat{\beta}'MA'(AMA')^{-1}b}{\hat{\beta}'M\hat{\beta} - \hat{\beta}'MA'(AMA')^{-1}AM\hat{\beta}} . \end{aligned}$$

Hence

$$\lambda = \frac{(\hat{\beta}' M \hat{\beta})b - c A M \hat{\beta}}{\hat{\beta}' M \hat{\beta} - \hat{\beta}' M A' (A M A')^{-1} A M \hat{\beta}} .$$

Similarly, the AB function has the form

$$(2.7) \quad \hat{x} = \frac{M}{\hat{\beta}' M \hat{\beta} + \sigma^2} (A' \lambda + \mu \hat{\beta}) .$$

A special example of such a restriction is the case when a constant term is included in the model, i.e.

$$(2.8) \quad E(Y) = \beta' x = \beta_0 + \beta_1' x_1 ,$$

where β_0 is a scalar. In this case, we can write

$$x = \begin{pmatrix} 1 \\ x_1 \end{pmatrix} ,$$

i.e., the first component of x is always equal to 1.

We shall redefine the parameter as

$$(2.9) \quad E(Y) = \tilde{\beta}_0 + \tilde{\beta}_1' \tilde{x}_1 ,$$

so that for the sample values the mean vector of $\tilde{x}_1$ is equal to zero. Then

$$M = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{M} \end{pmatrix} ,$$

where $\tilde{M}$ is the moment matrix of $\tilde{x}_1$, or the matrix of the second order moment of x_1 around the mean.

For simplicity of notation, we shall omit the tilde of (2.9) and it immediately follows that the ABC solution is determined so as to minimize $x_1' \tilde{M}^{-1} x_1$ under the condition that $\hat{\beta}_0 + \hat{\beta}_1' x_1 = c$, i.e.,

$$(2.10) \quad \hat{x}_1 = \frac{(c - \bar{Y}) \tilde{M} \hat{\beta}_1}{\hat{\beta}_1' \tilde{M} \hat{\beta}_1},$$

where $\bar{Y} = \hat{\beta}_0$ is the mean of Y values in the sample. If the mean vector of x_1 is not equal to 0, then $\hat{x}_1$ in (2.10) should be replaced for $\hat{x}_1 - \bar{x}_1$, where $\bar{x}_1$ is the mean vector.

Similarly, for the AB function, x_1 is determined as

$$(2.11) \quad x_1 = x_1 + \frac{(c - \bar{Y}) \tilde{M} \hat{\beta}_1}{\hat{\beta}_1' \tilde{M} \hat{\beta}_1 + \hat{\sigma}^2}$$

3. Exact Formula for the First Two Moments of the Error

So far the justification for particular functions is given in asymptotic terms. In this section we shall consider the finite sample property of the procedures of the previous section.

We shall consider the following quantity

$$(3.1) \quad \hat{x} = \frac{cM\hat{\beta}}{\hat{\beta}'M\hat{\beta} + a^2}$$

for an ABC function $a = 0$ in (3.1) and for an AB function $a^2 = \hat{\sigma}^2$. More generally, we shall consider the case where $a^2 = k\hat{\sigma}^2$ and k is a positive constant.

Without loss of generality, we can assume that $c = 1$, and we consider the first two moments of the error, i.e.,

$$E(\beta'\hat{x} - 1) \quad \text{and} \quad E(\beta'\hat{x} - 1)^2.$$

For the moment we shall assume that a is a constant, which corresponds to the case when σ^2 is known. Under the normality assumption, it is well known that $\beta'M\hat{\beta}$ is distributed normally with mean $\beta'M\beta$ and variance $\beta'M\beta\sigma^2$. Also $\hat{\beta}'M\hat{\beta}/\sigma^2$ is distributed according to a non-central chi-square distribution with p degrees of freedom and with the noncentrality parameter equal to $\tau = \beta'M\beta/\sigma^2$.

Let us define $\phi^2 = \beta'M\beta$ and

$$Z = \frac{\beta'M\hat{\beta}}{\phi} \quad \text{and} \quad Y = \hat{\beta}'M\hat{\beta} - Z^2.$$

Then it can easily be shown that Z is normally distributed

with mean ϕ and variance σ^2 , and Y/σ^2 is distributed independently of Z according to a chi-square distribution with $p-1$ degrees of freedom (assuming $p \geq 2$). Hence,

$$(3.2) \quad E\left(\frac{\beta' \hat{x}}{\phi}\right) = E\left(\frac{Z}{Y + Z^2 + a^2}\right) \\ = \int_0^\infty \int_{-\infty}^\infty \frac{z}{y + z^2 + a^2} \frac{1}{\sqrt{2\pi} \Gamma(\frac{p-1}{2}) \sigma^r} \left(\frac{y}{2}\right)^{\frac{p-3}{2}} e^{-\frac{y+(z-\phi)^2}{2\sigma^2}} dy dz.$$

In order to calculate this value, we shall define a function $H_{a,b}(s)$ of $s > 0$ by

$$(3.3) \quad H_{a,b}(s) = \int_0^\infty \int_{-\infty}^\infty \frac{z}{y + z^2 + a^2} \frac{1}{\sqrt{2\pi} \Gamma(\frac{p-1}{2})} \left(\frac{y}{2}\right)^{\frac{p-3}{2}} e^{-\frac{(y+z^2)}{2s} + bz} dy dz.$$

Then

$$(3.4) \quad E(\beta' \hat{x}) = \frac{\phi e^{-\frac{\phi^2}{2\sigma^2}}}{\sigma^p} H_{a,\phi/\sigma^2}(\sigma^2).$$

Differentiating (3.3) with respect to s (it is easy to see that differentiation under the integral sign is permitted in this case), we have

$$\begin{aligned} \frac{d}{ds} H_{a,b}(s) &= \iint \frac{z(y+z^2)}{2s^2(y+z^2+a^2)} \frac{1}{\sqrt{2\pi} \Gamma(\frac{p-1}{2})} \left(\frac{y}{2}\right)^{\frac{p-3}{2}} e^{-\frac{y+z^2}{2s} + bz} dy dz \\ &= \frac{1}{2s^2} \iint \frac{1}{\sqrt{2\pi} \Gamma(\frac{p-1}{2})} z \left(\frac{y}{2}\right)^{\frac{p-3}{2}} e^{-\frac{y+z^2}{2s} + bz} dy dz - \frac{a^2}{2s^2} H_{a,b}(s) \\ &= \frac{b s^{p/2-1}}{2} e^{b^2 s/2} - \frac{a^2}{2s^2} H_{a,b}(s). \end{aligned}$$

Thus we arrive at a differential equation

$$(3.5) \quad H'_{a,b(s)} + \frac{d^2}{ds^2} H_{a,b(s)} = \frac{bs^{p/2-1}}{2} e^{b^2s/2}.$$

The solution of (3.5) is easily shown to be

$$H_{a,b(s)} = e^{a^2/2s} \left[\int \frac{bs^{p/2-1}}{2} e^{b^2s/2-a^2/2s} ds + c \right],$$

where c is the integration constant.

Since $H_{a,b(s)} \rightarrow 0$ as $s \rightarrow 0$, we have

$$(3.6) \quad H_{a,b(s)} = e^{a^2/2s} \int_0^s \frac{bt^{p/2-1}}{2} e^{b^2t/2-a^2/2t} dt.$$

Finally, in view of (3.4), we have

$$(3.7) \quad \begin{aligned} E(\beta' \hat{x}) &= \frac{\phi^2 e^{-\frac{\phi^2-a^2}{2\sigma^2}}}{\sigma^{p+2}} \int_0^{\sigma^2} \left(\frac{t}{2}\right)^{p/2-1} e^{-\frac{\phi^2}{2\sigma^2}t - \frac{a^2}{2t}} dt \\ &= \frac{\tau^{-\frac{p}{2}+1}}{2} e^{\frac{\alpha-\tau}{2}} \int_0^{\tau} t^{p/2-1} e^{\frac{t}{2} - \frac{\alpha\tau}{2t}} dt, \end{aligned}$$

where $\tau = \phi^2/\sigma^2$, $\alpha = a^2/\sigma^2$.

It is easy to see that formula (3.7) can be applied also for the case $p = 1$ provided $a > 0$, and if $p = 1$ and $a = 0$,

$$E(\beta' x) = \phi E\left(\frac{1}{Z}\right)$$

does not exist. Since the expression can be written as

$$(3.8) \quad E(\beta' \hat{x}) = \frac{\tau^{-p/2+1}}{2} e^{-\tau/2} \int_0^{\tau} t^{p/2-1} e^{\frac{t}{2} - \frac{\alpha}{2}(\frac{\tau}{t}-1)} dt,$$

$E(\beta' \hat{x})$ is monotone decreasing in α ($\alpha \geq 0$).

When $p \geq 2$, and $\alpha = 0$, we have

$$(3.9) \quad E(\beta' \hat{x}) = \frac{\tau^{-p/2+1}}{2} e^{-\tau/2} \int_0^{\tau} t^{p/2-1} e^{-t/2} dt,$$

which is easily shown to be monotone increasing in τ , to tend to zero when τ tends to zero, and to tend to one as τ tends to infinity.

Hence, it is established that

$$E(\beta' \hat{x}) - 1$$

is always negative, monotone increasing in τ and hence in ϕ^2 , and decreasing in α or in a^2 . Thus the bias of the controlled level is always negative, and the absolute value of the bias is smaller for ABC than for AB, or for Bayes procedures.

When $p = 1$, (3.9) can be regarded as the limit of $E(\beta' \hat{x})$ as $\alpha \rightarrow 0$. Hence, similar results hold true also in this case only with slight modifications.

As for the second order moment

$$\begin{aligned} \frac{E(\beta' \hat{x})^2}{\phi^2} &= E\left(\frac{Z}{Y+Z^2+a^2}\right)^2 \\ &= \int_0^{\infty} \int_{-\infty}^{\infty} \frac{z^2}{(y+z^2+a^2)^2} \frac{1}{\sqrt{2\pi} \Gamma(\frac{p-1}{2}) \sigma^p} \left(\frac{y}{2}\right)^{\frac{p-3}{2}} e^{-\frac{y+(z-\phi)^2}{2\sigma^2}} dy dz \end{aligned}$$

We shall define

$$(3.10) \quad G_{a,b}(s) = \int_0^\infty \int_{-\infty}^\infty \frac{z^2}{(y+z^2+a^2)^2} \frac{1}{\sqrt{2\pi} \Gamma(\frac{p-1}{2})} \left(\frac{y}{2}\right)^{\frac{p-3}{2}} \\ \cdot e^{-\frac{s(y+z^2+a^2)}{2} + bz} dy dz .$$

Then

$$(3.11) \quad E(\beta, \hat{x})^2 = \frac{\phi^2 e^{-\frac{\phi^2}{2\sigma^2} + \frac{a^2}{2\sigma^2}}}{\sigma^p} G_{a,\phi/\sigma^2}(\sigma^{-2}) .$$

Differentiating (3.10) twice with respect to s , we have

$$(3.12) \quad \frac{d^2}{ds^2} G_{a,b}(s) = \frac{1}{4} \int_0^\infty \int_{-\infty}^\infty z^2 \frac{1}{\sqrt{2\pi} \Gamma(\frac{p-1}{2})} \left(\frac{y}{2}\right)^{\frac{p-3}{2}} \\ = \frac{1}{4} \left(\frac{b^2}{s^2} + \frac{1}{s}\right) s^{-\frac{p}{2}-2} e^{\frac{b^2}{2s} - \frac{a^2 s}{2}} .$$

Hence, if we put

$$Q(s) = \int (b^2 + s) s^{-\frac{p}{2}-2} e^{\frac{b^2}{2s} - \frac{a^2 s}{2}} ds ,$$

we have

$$\begin{aligned} 4 G_{a,b}(s) &= \int Q(s) ds + c_1 s + c_2 s \\ &= s Q(s) - \int s Q'(s) ds + c_1 s + c_2 s \\ &= s \int (b^2 + s) s^{-\frac{p}{2}-2} e^{\frac{b^2}{2s} - \frac{a^2 s}{2}} ds - \end{aligned}$$

$$- \int (b^2 + s)s^{-\frac{p}{2}-1} e^{\frac{b^2}{2s} - \frac{a^2 s}{2}} ds + c_1 s + c_2 s, \quad$$

where c_1 and c_2 are integration constants. Finally it can be shown (considering the limiting case as $s \rightarrow \infty$ and $s \rightarrow 0$) that

$$\begin{aligned} {}_4G_{a,b}(s) &= \text{const.} - \int_s^\infty (s-t)(b^2+t)t^{-\frac{p}{2}-2} e^{\frac{b^2}{2t} - \frac{a^2 t}{2}} dt \\ &= b^2 s \int_0^{1/s} \left(\frac{1}{s} - t\right) \left(\frac{1}{b^2} + t\right) t^{\frac{p}{2}-2} e^{\frac{b^2 t}{2} - \frac{a^2}{2t}} dt. \end{aligned}$$

Thus, in view of (3.11), we have

$$\begin{aligned} E(\beta' \hat{x})^2 &= \frac{\phi^4}{4\sigma^{p+6}} e^{-\frac{p^2}{2\sigma^2} + \frac{a^2}{2\sigma^2}} \int_0^{\sigma^2} (\sigma^2 - t) \left(\frac{\sigma^4}{\phi^2} + t\right) t^{\frac{p}{2}-2} e^{\frac{\phi^2 t}{2\sigma^4} - \frac{a^2}{2t}} dt \\ (3.13) \quad &= \frac{-\frac{p}{2} + 1}{4} e^{-\frac{\tau - \alpha}{2}} \int_0^\tau (\tau - t)(1+t)t^{\frac{p}{2}-2} e^{\frac{t}{2} - \frac{\alpha\tau}{2t}} dt, \end{aligned}$$

where $\tau = \phi^2/\sigma^2$, $\alpha = a^2/\sigma^2$ as in (3.7).

The integral in (3.13) is monotone decreasing in α which can be shown quite similarly as in (3.7). Equation (3.13) holds true for all p if $\alpha > 0$, and for $p \geq 3$ if $\alpha = 0$, and if $p \leq 2$, $E(\beta' \hat{x})^2$ tends to infinity as $\alpha \rightarrow 0$.

Thus for the mean square

$$E(\beta' \hat{x} - 1)^2 = \{E(\beta' \hat{x})^2 - 1\} - 2\{E(\beta' \hat{x}) - 1\},$$

the first term decreases while the second term increases as α increases.

As for the more general situation where $a = c\hat{\sigma}^2$, where $\hat{\sigma}^2$ is an estimator of σ^2 and independent of $\hat{\beta}$, (3.7) and (3.13) can be regarded as formulae for the conditional moments given $\hat{\sigma}^2$, hence unconditional moments can be given by taking expectations of these with respect to $\hat{\sigma}^2$.

Assume that $\hat{\sigma}^2 = W\sigma^2/q$, where W is a chi-square variable with q degrees of freedom and is independent of $\hat{\beta}$. Assume that $a^2 = k\hat{\sigma}^2$ or $\alpha = kW/q$, where $k \geq 0$ is a constant.

Then, from (3.7), the unconditional expectation of $\beta'\hat{x}$ is given by

$$\begin{aligned}
 E(\beta'\hat{x}) &= \frac{\tau^{-p/2+1}}{2} e^{-\tau/2} \int_0^\tau \int_0^\infty t^{p/2-1} e^{t/2} e^{-k(\tau-t)w/2qt} \\
 &\quad \cdot \frac{1}{2\Gamma(\frac{q}{2})} \left(\frac{w}{2}\right)^{q/2-1} e^{-w/2} dt dw \\
 (3.14) \quad &= \frac{\tau^{-p/2+1}}{2} e^{-\tau/2} \int_0^\tau \frac{t^{p/2-1}}{(1 + \frac{k(\tau-t)}{qt})^{q/2}} e^{t/2} dt \\
 &= \frac{\tau^{-p/2+1}}{2} e^{-\tau/2} \int_0^\tau \frac{t^{\frac{p+q}{2}-1}}{(t + \frac{k(\tau-t)}{q})^{q/2}} e^{t/2} dt,
 \end{aligned}$$

provided that either $p \geq 2$ or $k > 0$.

Similarly, from (3.13),

$$(3.15) \quad E(\beta'\hat{x})^2 = \frac{\tau^{-p/2+1}}{4} e^{-\tau/2} \int_0^\tau \frac{(\tau-t)(1+t)t^{\frac{p+q}{2}-2}}{(t + \frac{k(\tau-t)}{q})^{q/2}} e^{t/2} dt,$$

assuming $p \geq 3$ or $k > 0$.

4. Asymptotic Expansion when τ is Large.

Generally, $\tau = \beta' M \beta / \sigma^2$ can be expected to be large, so that it is worthwhile to consider the limiting case when τ is large.

Assuming τ is large,

$$\begin{aligned}
 E(\beta' \hat{x}) &= \frac{1}{2} \int_0^\tau \frac{\left(\frac{t}{\tau}\right)^{\frac{p+q}{2}-1}}{\left(\frac{t}{\tau} + \frac{k(\tau-t)}{q\tau}\right)^{q/2}} e^{-\frac{\tau-t}{2}} dt \\
 &= \frac{1}{2} \int_0^\tau \frac{\left(1 - \frac{t}{\tau}\right)^{\frac{p+q}{2}-1}}{\left(1 - d \frac{t}{\tau}\right)^{q/2}} e^{-t/2} dt \quad \text{where } d = 1 - \frac{k}{q} \\
 &= \frac{1}{2} \int_0^\tau \left\{ 1 - (p+q-2) \frac{t}{2\tau} + (p+q-2)(p+q-4) \frac{t^2}{8\tau^2} + \dots \right\} \\
 &\quad \times \left\{ 1 + q d \frac{t}{2\tau} + q(q+2)d^2 \frac{t^2}{8\tau^2} + \dots \right\} e^{-t/2} dt \\
 &= \frac{1}{2} \int_0^\tau \left[1 - (p+k-2) \frac{t}{2\tau} + \{(p+q-2)(p+q-4) - 2qd(p+q-2) \right. \\
 &\quad \left. + q(q+2)d^2\} \frac{t^2}{8\tau^2} + \dots \right] e^{-t/2} dt \\
 &= 1 - \frac{1}{\tau} (p+k-2) + \frac{1}{\tau^2} \{(p+k-2)(p+k-4) - 2k(1 - \frac{k}{q})\} + o(\frac{1}{\tau^2}).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
E(\beta' \hat{x})^2 &= \frac{1}{4} \int_0^{\tau} \frac{t(1 + \frac{1}{\tau} - \frac{t}{\tau})(1 - \frac{t}{\tau})^{\frac{p+q}{2} - 2}}{(1 - \frac{dt}{\tau})^{q/2}} e^{-t/2} dt \\
&= \frac{1}{4} \int_0^{\tau} t(1 + \frac{1}{\tau} - \frac{t}{\tau}) \{1 - (p+q-4)\frac{t}{2\tau} + (p+q-4)(p+q-6)\frac{t^2}{8\tau^2} + \dots\} \\
&\quad \times \{1 + q \frac{dt}{2\tau} + q(q+2)d^2 \frac{t^2}{8\tau^2} + \dots\} e^{-t/2} dt \\
&= \frac{1}{4} \int_0^{\tau} [t + \frac{t(1-t)}{\tau} - (p+k-4) \frac{t^2}{2\tau} - (p+k-4) \frac{t^2(1-t)}{2\tau^2} \\
&\quad + \{(p+q-4)(p+q-6) - 2q \frac{d}{dt}(p+q-4) + q(q+2) d^2\} \frac{t^3}{8\tau^2} + \dots] e^{-t/2} dt \\
&= 1 - \frac{3}{\tau} - 2(p+k-4) \frac{1}{\tau} + 10(p+k-4) \frac{1}{\tau^2} \\
&\quad + \frac{3}{\tau^2} \{(p+k-4)(p+k-6) - 2k(1 - \frac{k}{q})\} + o(\frac{1}{\tau^2}) \\
&= 1 - \frac{1}{\tau} \{2(p+k) - 5\} + \frac{3}{\tau^2} \{(p+k-4)(p+k - \frac{8}{3}) - 2k(1 - \frac{k}{q})\} + o(\frac{1}{\tau^2}).
\end{aligned}$$

Hence, we have finally

$$\begin{aligned}
E(\beta' \hat{x} - 1)^2 &= E(\beta' \hat{x})^2 - 2 E(\beta' \hat{x}) + 1 \\
(4.1) \qquad &= \frac{1}{\tau} + \frac{1}{\tau^2} \{ (p+k-4)^2 - 2k(1 - \frac{k}{q}) \} + o(\frac{1}{\tau^2}) ,
\end{aligned}$$

provided either $p \geq 3$ or $k > 0$.

It should be remarked that in (4.1) the first term is independent of all the parameters p, q, k , and actually equal to the asymptotic value given in (2.3). Hence, at least when τ is large enough, the choice of k does not affect the situation very much provided only that when $p = 1$ or 2 , k must not be put to 0 .

The second term is minimized when

$$(4.2) \qquad k = \max \left(\frac{5-p}{1 + \frac{2}{q}}, 0 \right) .$$

Hence, we can recommend as at least asymptotically good strategy to adopt the value of k given by (4.2) or, assuming q is large, a simpler one, i.e.,

$$k = \max (5 - p , 0) .$$

When the constant term is included in the model, it is necessary from (2.10) and (2.11) that we consider a function of the type

$$(4.4) \qquad \hat{x} = \bar{x} + (c - \bar{y}) \frac{\tilde{M} \hat{\beta}}{\hat{\beta}' \tilde{M} \hat{\beta} + k \hat{\sigma}^2} ,$$

where $\tilde{M}$ is the moment matrix of x values around the mean.

Then under the assumptions of normality, $\bar{y}$ and $\hat{\beta}$, $\hat{\sigma}^2$ are independent, and it is shown that

$$(4.5) \quad E(\beta' \hat{x} - c) = (\beta' \bar{x} - c) E\left(\frac{\beta' \tilde{M} \hat{\beta}}{\hat{\beta}' \tilde{M} \hat{\beta} + k \hat{\sigma}^2} - 1\right);$$

$$(4.6) \quad E(\beta' \hat{x} - c)^2 = (\beta' \bar{x} - c)^2 E\left(\frac{\beta' \tilde{M} \hat{\beta}}{\hat{\beta}' \tilde{M} \hat{\beta} + k \hat{\sigma}^2} - 1\right) + \frac{\sigma^2}{n} E\left(\frac{\beta' \tilde{M} \hat{\beta}}{\hat{\beta}' \tilde{M} \hat{\beta} + k \hat{\sigma}^2}\right)^2;$$

and the moments of $\beta' \tilde{M} \hat{\beta} / (\hat{\beta}' \tilde{M} \hat{\beta} + k \hat{\sigma}^2)$ are given from the same formulae thus far obtained only by substituting $\beta' \tilde{M} \beta / \sigma^2$ for τ . Hence, the only difference which matters comes from the second term of (4.6), which we can assume, however, to be nearly equal to σ^2/n when τ is large, and the influence of k to this term may be negligible when n is large.

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